3.1. One assumes

$$\langle \delta(x - x_t) e^{-\beta W(t)} \rangle = \frac{e^{-\beta H(x, \lambda_t)}}{Z_0}$$

Integrate over x and choose t to be final time, then:

$$\langle e^{-\beta W} \rangle = \frac{1}{Z_0} \int e^{-\beta H(x, \lambda_t)} dx = \frac{Z_{\lambda_t}}{Z_0} = e^{-\beta \Delta F}$$

3.2 $$\langle \delta(x - x_t)\, e^{-\beta W(t)} \rangle = \langle e^{-\beta W(t)} \,|\, x = x_t \rangle \underbrace{\langle \delta(x - x_t) \rangle}_{\rho(x, t)}$$

$$= \frac{e^{-\beta H(x, \lambda_t)}}{Z_0}$$

Now dividing by $\rho_{eq}(x, \lambda_t) = \frac{e^{-\beta H(x, \lambda_t)}}{Z_t}$, one obtains

$$\frac{\rho(x, t)}{\rho_{eq}(x, \lambda_t)} \langle e^{-\beta W(t)} \,|\, x = x_t \rangle = \frac{e^{-\beta \cancel{H(x, \lambda_t)}}}{Z_0} \frac{Z_t}{e^{-\beta \cancel{H(x, \lambda_t)}}} =$$

$$= e^{-\beta \Delta F(t)}$$

where $\Delta F(t) = F_{\lambda_t} - F_{\lambda_0}$ is the free energy difference between a state with a control parameter λ_t and a state with a control parameter λ_0.

Rq it is important to appreciate that at time t, the system may not be in equilibrium although the equilibrium free energy shows up in Jarzynski relation and is evaluated at $\lambda = \lambda_t$.

3.3 The free energy F and internal energy U are state functions, therefore their variation on a cycle is zero

$$\Delta F = \Delta U = 0$$

3.4 One can use the previous relation evaluated at the end of the protocol $t = \tau$:

$$\langle e^{-\beta W(t)} \mid x = x_\tau \rangle = \frac{\rho_{eq}(x, \lambda_\tau)}{\rho(x, \tau)}$$

If one considers only trajectories ending in the right minimum at time τ, then $\rho(x = x_R, \tau) = 1$ and $\rho_{eq}(x, \lambda_\tau) = 1/2$ given the initial condition (left or right with equal proba)

Thus $\langle e^{-\beta W} \rangle_{\to 0} = \frac{1}{2}$

Similarly one gets $\langle e^{-\beta W} \rangle_{\to 1} = \frac{1}{2}$

3.5 If the process is not perfect, we have instead

$\rho(x = x_R, \tau) = P_s$, then

$$\langle e^{-\beta W} \rangle_{\to 0} = \frac{1}{2 P_s} \quad \text{and} \quad \langle e^{-\beta W} \rangle_{\to 1} = \frac{1}{2(1-P_s)}$$

Then taken together, the usual Jarzynski relation is recovered

$$\langle e^{-\beta W} \rangle = P_s \langle e^{-\beta W} \rangle_{\to 0} + (1-P_s) \langle e^{-\beta W} \rangle_{\to 1}$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

3.6 Using Jensen's inequality $\langle e^x \rangle \geqslant e^{\langle x \rangle}$, one obtains

$$\beta \langle W \rangle \geqslant \ln 2 + P_s \ln P_s + (1-P_s) \ln (1-P_s)$$

Using general expression of Shanon entropy $S = -k \sum_i p_i \ln p_i$ for a probability distribution p_i, it follows that

$$S_{t=0} = -k\left(\frac{1}{2} \ln \frac{1}{2} + \frac{1}{2} \ln \frac{1}{2}\right) = k \ln 2$$

$$S_{t=\tau} = -k\left(P_s \ln P_s + (1-P_s) \ln (1-P_s)\right)$$

thus $\Delta S = k\left[\ln 2 + P_s \ln P_s + (1-P_s) \ln (1-P_s)\right]$

and $\beta \langle W \rangle \geqslant \frac{\Delta S}{k}$

or $\langle W \rangle \geqslant T \Delta S$

3,7 The figure shows that in the quasi-static limit $\tau \to 0$, $\langle Q \rangle \to kT \ln 2$.

Since $\langle \Delta U \rangle = 0 = \langle W \rangle - \langle Q \rangle$, this means $\langle W \rangle \to kT \ln 2$ which corresponds to the expected limit when $P_s \to 1$.

The process corresponds to a logically irreversible operation because the output (a particle always in the right well) does not permit to predict the input (a particle in the right or left).